

Closing Wed: HW_9A, 9B (9.3, 9.4)
 Final Exam, Saturday, March 10th
 Kane 210, 1:30-4:20pm

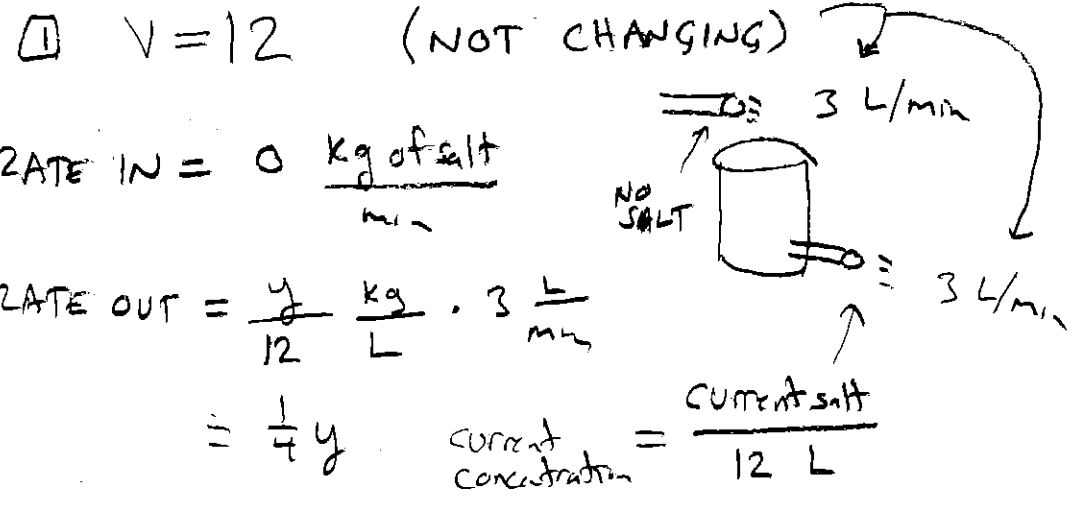
9.4 Diff. Eq. Apps (continued)

Entry Task: Assume a 12 Liter vat contains 7 kg of salt initially. A pipe pumps in *pure water* at 3 L/min. The vat is well mixed. The mixture drains at 3 L/min.

Let $y(t)$ = "kg of salt in vat at time t ".

Identify and label the following:

1. Volume of the vat (Is it changing?)
2. Amount of salt per min entering.
3. Amount of salt per min exiting.
4. Initial amount of salt.



④ $y(0) = 7$

$$\frac{dy}{dt} = 0 - \frac{1}{4}y, \quad y(0) = 7$$

$$\int \frac{1}{y} dy = \int -\frac{1}{4} dt$$

$$\ln|y| = -\frac{1}{4}t + C_1$$

$$|y| = e^{(-\frac{1}{4}t + C_1)}$$

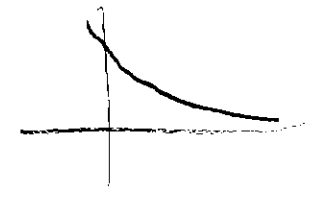
$$y(t) = \pm e^{C_1} e^{-\frac{1}{4}t}$$

$$C = \pm e^{C_1}$$

$$y(t) = C e^{-\frac{1}{4}t}$$

$$y(0) = 7 \Rightarrow C = 7$$

$$y(t) = 7 e^{-\frac{1}{4}t}$$



$$\boxed{\text{RATE IN}} \quad 2 \frac{\text{kg}}{\text{L}} \cdot 3 \frac{\text{L}}{\text{min}} = 6 \frac{\text{kg of salt}}{\text{min}}$$

$$\boxed{\text{RATE OUT}} \quad \frac{y}{12} \frac{\text{kg}}{\text{L}} \cdot 3 \frac{\text{L}}{\text{min}} = \frac{3y}{12} = \frac{y}{4} \frac{\text{kg}}{\text{min}}$$

$$\frac{dy}{dt} = 6 - \frac{1}{4}y, \quad y(0) = 7$$

$$\int \frac{1}{6 - \frac{1}{4}y} dy = \int dt$$

$$-4 \ln|6 - \frac{1}{4}y| = t + C_1$$

$$C_2 = -\frac{1}{4}C_1$$

$$\ln|6 - \frac{1}{4}y| = -\frac{1}{4}t + C_2$$

$$|6 - \frac{1}{4}y| = e^{C_2 - \frac{1}{4}t}$$

$$6 - \frac{1}{4}y = \pm e^{C_2} e^{-\frac{1}{4}t}$$

$$C_3 = \pm e^{C_2}$$

$$6 = C_3 e^{-\frac{1}{4}t} + \frac{1}{4}y$$

$$\frac{1}{4}y = 6 - C_3 e^{-\frac{1}{4}t}$$

$$C = -4C_3$$

$$\boxed{y = 24 + C e^{-\frac{1}{4}t}}$$

$$y(0) = 7 \Rightarrow 7 = 24 + C \Rightarrow C = -17$$

$$\boxed{y(t) = 24 - 17e^{-\frac{1}{4}t}}$$

$$\boxed{\lim_{t \rightarrow \infty} y(t) = 24}$$

Example:

Assume a 12 Liter vat contains 7 kg of salt initially. A pipe pumps in salt water (brine) at 3 L/min with a concentration of 2 kg/L of salt.

The vat is well mixed.

The mixture leaves the vat at 3 L/min.

Let $y(t)$ = the amount of salt in the vat at time t .

(a) Find $y(t)$.

(b) Find the limit of $y(t)$ as $n \rightarrow \infty$.

Here is what these problems typically look like:

V = volume of vat (liters)

t = time (min)

$y(t)$ = amount in vat (kg)

$\frac{dy}{dt}$ = rate (kg/min)

Thus,

$$\begin{aligned} \frac{dy}{dt} &= \text{Rate In} - \text{Rate out} \\ &= \left(? \frac{\text{kg}}{\text{L}} \right) \left(? \frac{\text{L}}{\text{min}} \right) - \left(\frac{y}{V} \frac{\text{kg}}{\text{L}} \right) \left(? \frac{\text{L}}{\text{min}} \right) \end{aligned}$$

$$y(0) = ? \text{ kg}$$

Example:

Assume a 100 Liter vat contains 5kg of salt initially. Two pipes (A & B) pump in salt water (brine).

Pipe A: Enters at 3L/min with a concentration of 4kg/L of salt.

Pipe B: Enters at 5L/min with a concentration of 2kg/L of salt.

The vat is well mixed.

The mixture leaves the vat at 8L/min.

Let $y(t)$ = the amount of salt in the vat at time t .

(a) Find $y(t)$.

(b) Find the limit of $y(t)$ as $n \rightarrow \infty$.

$$\boxed{\text{RATE IN}} \quad \underbrace{\frac{4 \text{ kg}}{\text{L}} \cdot \frac{3 \text{ L}}{\text{min}}}_{12} + \underbrace{\frac{2 \text{ kg}}{\text{L}} \cdot \frac{5 \text{ L}}{\text{min}}}_{10} = 22 \frac{\text{kg of salt}}{\text{min}}$$

$$\boxed{\text{RATE OUT}} \quad \frac{y}{100} \cdot \frac{8 \text{ L}}{\text{min}} = \frac{8y}{100} = \frac{2y}{25} \frac{\text{kg}}{\text{min}}$$

$$\frac{dy}{dt} = 22 - \frac{2y}{25}, \quad y(0) = 5$$

$$\int \frac{1}{22 - \frac{2y}{25}} dy = \int dt$$

$$-\frac{25}{2} \ln \left| 22 - \frac{2y}{25} \right| = t + C_1$$

$$\ln \left| 22 - \frac{2y}{25} \right| = -\frac{2}{25} t + C_2$$

$$22 - \frac{2y}{25} = \pm e^{C_2} e^{-\frac{2}{25} t}$$

$$22 - \frac{2y}{25} = C_3 e^{-\frac{2}{25} t}$$

$$\frac{2y}{25} = 22 - C_3 e^{-\frac{2}{25} t}$$

$$C_2 = -\frac{25}{2} C_1$$

$$C_3 = \pm e^{C_2}$$

$$C = -\frac{25}{2} C_3$$

$$\boxed{y = 275 + C e^{-\frac{2}{25} t}}$$

$$y(0) = 5 \Rightarrow C = -270$$

$$\boxed{\lim_{t \rightarrow \infty} y(t) = 275 \text{ kg}}$$

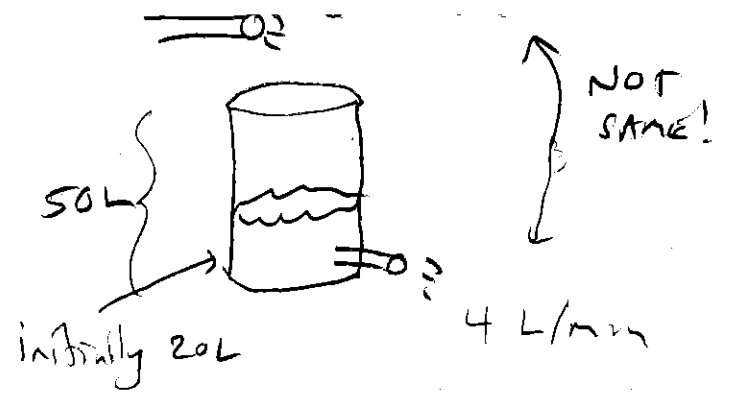
Example: Assume a 50 Liter container currently has 20 Liters of water with 24 kg of dissolved salt.

A pipe pumps in *pure water* at 6 L/min. The vat is well mixed.

The mixture drains at 4 L/min.

Let $y(t)$ = "kg of salt in vat at time t ".

What is different about this problem?



$$\text{VOLUME} = V(t) = 20 + 2t$$

why?
 $6 - 4 = 2 \frac{\text{L}}{\text{min}}$
 added

$$\text{RATE IN} = 0 \frac{\text{kg}}{\text{L}} \cdot 6 \frac{\text{L}}{\text{min}} = 0 \frac{\text{kg}}{\text{min}}$$

$$\text{RATE OUT} = \frac{y}{20+2t} \frac{\text{kg}}{\text{L}} \cdot 4 \frac{\text{L}}{\text{min}} = \frac{4y}{20+2t} = \frac{2y}{10+t} \frac{\text{kg}}{\text{L}}$$

$$\frac{dy}{dt} = 0 - \frac{2y}{10+t}, \quad y(0) = 24$$

$$\Rightarrow \int \frac{1}{y} dy = \int -\frac{2}{10+t} dt$$

$$\ln|y| = -2 \ln|10+t| + C_1$$

$$\Rightarrow |y| = e^{(-2 \ln(10+t) + C_1)}$$

$$y = \pm e^{-2 \ln(10+t)} C_1$$

$$y = C e^{-2 \ln(10+t)}$$

$$y = C (10+t)^{-2}$$

$$y = \frac{C}{(10+t)^2}$$

NOTE
 $-2 \ln(10+t)$
 $= \ln((10+t)^{-2})$
 $\ln(10+t)^4$
 $e^{\ln(10+t)^4} = (10+t)^4$

$$y(0) = 24 \Rightarrow \frac{C}{10^2} = 24$$

$$y(t) = \frac{2400}{(10+t)^2}$$

$$C = 2400$$

4. Air Resistance:

A skydiver steps out of a plane that is 4,000 meters high with an initial downward velocity of 0 m/s.

The skydiver has a mass of 60 kg. (Treat downward as positive).

Let $y(t)$ = "height at time t "

Newton's 2nd Law says:

(mass)(acceleration) = Force

$$m \frac{d^2 y}{dt^2} = \text{sum of forces on the object}$$

The force due to gravity has constant magnitude (acting downward):

$$F_g = mg = 60 \cdot 9.8 = 588 \text{ N}$$

TERMINAL VELOCITY
↓

$$\lim_{t \rightarrow \infty} v(t) = 49 \text{ m/sec}$$

One model for air resistance

The force due to air resistance (*drag force*) is proportional to velocity and in the opposite direction of velocity. So

$$F_d = -k v \text{ Newtons}$$

Assume for this problem $k = 12$.

$$m \frac{dv}{dt} = mg - kv$$

$$\int \frac{m}{mg - kv} dv = \int dt$$

$$-\frac{m}{k} \ln |mg - kv| = t + C_1$$

$$\ln |mg - kv| = -\frac{k}{m} t - \frac{k}{m} C_1 \quad C_2 = -\frac{k}{m} C_1$$

$$mg - kv = \pm e^{C_2} e^{-\frac{k}{m} t} \quad C_3 = \pm e^{C_2}$$

$$kv = mg - C_3 e^{-\frac{k}{m} t} \quad C = -\frac{1}{k} C_3$$

$$v(t) = \frac{mg}{k} + C e^{-\frac{k}{m} t}$$

$$v(t) = \frac{588}{12} + C e^{-\frac{12}{60} t} = 49 + C e^{-\frac{1}{5} t}$$

$$v(0) = 0 \Rightarrow C = -49$$

$$v(t) = 49 - 49 e^{-\frac{1}{5} t}$$

Spring 2011 Final:

$v(t)$ = velocity of an object

$$F = mg - kv$$

Recall:

$$F = ma = m \frac{dv}{dt}$$

You are given m , g , and k and asked
for solve for $v(t)$.

DOME!

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Spring 2014 Final:

A lake has a volume of 1000 cubic meters and contains pure water. A farm near the lake begins using a pesticide.

Runoff from the farmland into the lake is 10 cubic meters of water per day, with a concentration of 50 grams of pesticide per cubic meter of water. The lake drains to the ocean at a rate of 10 cubic meters per day.

Find the formula $p(t)$ for the amount of pesticide in the lake at time t days.

$$\boxed{\text{RATE IN}} \quad 50 \frac{\text{g}}{\text{m}^3} \cdot 10 \frac{\text{m}^3}{\text{day}} = 500 \frac{\text{grams}}{\text{day}}$$

$$\boxed{\text{RATE OUT}} \quad \frac{p}{1000} \frac{\text{g}}{\text{m}^3} \cdot 10 \frac{\text{m}^3}{\text{day}} = \frac{1}{100} p \frac{\text{grams}}{\text{day}}$$

$$\boxed{\frac{dp}{dt} = 500 - \frac{1}{100} p, \quad p(0) = 0}$$

$$\int \frac{1}{500 - \frac{1}{100} p} dp = \int dt$$

$$-100 \ln |500 - \frac{1}{100} p| = t + C_1$$

$$C_2 = -\frac{1}{100} C_1$$

$$\ln |500 - \frac{1}{100} p| = -\frac{1}{100} t + C_2$$

$$500 - \frac{1}{100} p = \pm e^{C_2} e^{-\frac{1}{100} t}$$

$$C_3 = \pm e^{C_2}$$

$$\frac{1}{100} p = 500 - C_3 e^{-\frac{1}{100} t}$$

$$\boxed{p = 50,000 + C_4 e^{-\frac{1}{100} t}}$$

$$C_4 = -100 C_3$$

$$p(0) = 0 \Rightarrow \boxed{p(t) = 50,000 - 50,000 e^{-\frac{1}{100} t}}$$

Winter 2011 Final:

Your friend wins the lottery, and gives you P_0 dollars to help you pay your college expenses. The money is invested in a savings account that earns 10% annual interest, compounded continuously, and you withdraw the money continuously at an average rate of \$3600 per year.

Find the formula $A(t)$ for the amount of money in the account after t years.

$$\frac{dA}{dt} = \begin{array}{l} \text{change in} \\ \text{dollars per year} \end{array}$$

$$\frac{dA}{dt} = \begin{array}{l} \text{INTEREST} \\ \text{ADDED} \\ \text{PER YEAR} \end{array} - \begin{array}{l} \text{WITHDRAWALS} \\ \text{PER YEAR} \end{array}$$

$$\frac{dA}{dt} = 0.1A - 3600, \quad A(0) = P_0$$

$$\int \frac{1}{0.1A - 3600} dA = \int dt$$

$$\frac{1}{0.1} \ln|0.1A - 3600| = t + C_1, \quad C_2 = \frac{1}{10} C_1$$

$$\ln|0.1A - 3600| = \frac{1}{10} t + C_2$$

$$0.1A - 3600 = \pm e^{C_2} e^{\frac{1}{10} t}, \quad C_3 = \pm e^{C_2}$$

$$0.1A = C_3 e^{\frac{1}{10} t} + 3600$$

$$A = C e^{\frac{1}{10} t} + 36,000, \quad C = 10C_3$$

$$A(0) = P_0 \Rightarrow C + 36000 = P_0$$

$$\Rightarrow C = P_0 - 36000$$

$$A(t) = (P_0 - 36000) e^{\frac{1}{10} t} + 36000$$

Fall 2009 Final:

The swine flu epidemic has been modeled by the Gompertz function, which is a solution of

$$\frac{dy}{dt} = 1.2 y (K - \ln(y)),$$

where $y(t)$ is the number of individuals (in thousands) in a large city that have been infected by time t , and K is a constant.

On July 9, 2009, 75 thousand individuals had been infected.

One month later, 190 thousand individuals had been infected.

Find the formula $y(t)$ for the number of people that are infected t months, July 9, 2009.

$$\lim_{t \rightarrow \infty} y(t) = e^K \approx 283.629$$

$$\int \frac{1}{y(k - \ln(y))} dy = \int 1.2 dt$$

$$- \int \frac{1}{u} du = 1.2t + C_1 \quad \begin{array}{l} u = k - \ln(y) \\ du = -\frac{1}{y} dy \end{array}$$

$$-\ln|u| = 1.2t + C_1$$

$$\ln|u| = -1.2t - C_1$$

$$u = \pm e^{-C_1} e^{-1.2t}$$

$$C = \pm e^{C_1}$$

$$k - \ln(y) = C e^{-1.2t}$$

$$\ln(y) = k - C e^{-1.2t}$$

$$y = e^{(k - C e^{-1.2t})}$$

$$y(0) = 75 \Rightarrow k - C = \ln(75)$$

$$y(1) = 190 \Rightarrow k - C e^{-1.2} = \ln(190)$$

$$C = k - \ln(75)$$

$$\Rightarrow k - (k - \ln(75)) e^{-1.2} = \ln(190)$$

$$\Rightarrow k - k e^{-1.2} + \ln(75) e^{-1.2} = \ln(190)$$

$$\Rightarrow k(1 - e^{-1.2}) = \ln(190) - \ln(75) e^{-1.2}$$

$$\Rightarrow k = \frac{\ln(190) - \ln(75) e^{-1.2}}{1 - e^{-1.2}} \approx 5.64767$$

Side Note on Population Modeling

The Logistics Equation

Consider a population scenario where there is a limit (capacity) to the size of the population.

Let $P(t)$ = population size at time t .

M = maximum population size.
(capacity)

We sometimes want a model that

- a. ...is like natural growth when $P(t)$ is significantly smaller than M ;
- b. ...levels off (with a slope approaching zero), then the population approaches M .

One such model is the so-called logistics equation

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{M} \right) \text{ with } P(0) = P_0$$